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The phenomenon of change of the meaning of mathematical objects due to the passage between their different representations: how other disciplines can be useful to the analysis

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Abstract

In this paper we will demonstrate a consequence at times manifest in the semiotic transformations involving the treatment and conversion of a semiotic representation whose sense derives from a shared practice. The shift from one representation of a mathematical object to another via transformations maintains the meaning of the object itself on the one hand, but on the other hand it can change its sense. This is demonstrated in detail through a specific example, while at the same time it is collocated within a broad theoretical framework that poses fundamental questions concerning mathematical objects, their meanings and their representations.

Episodes

In D'Amore (2006), D'Amore and Fandiño Pinilla (2007a, b), we have reported and discussed, exclusively from a semiotic structural point of view, episodes taken from classroom situations in which students are mathematics teachers in their initial training, engaged in facing representations problems. Some examples of the phenomenon have been given orally in Rhodes, on April 13th 2006, during a general conference **(**How the treatment or conversion changes the sense of mathematical objects) at the 5th MEDCONF2007 (Mediterranean Conference on Mathematics Education), 13-15 April 2007, Rhodes, Greece (D'Amore, 2007).

The task consisted in this: working in small groups the trainee teachers received a text written in natural language; such texts had to be transformed into algebraic language. Once they had come to the algebraic formulation, this was explained by the group and collectively discussed. Our duty as university teachers was to suggest the further transformation of the obtained algebraic expressions into other algebraic *expressions*, to face collective discussions on their meaning.

We present three examples below:

Example 1
[We omit the original linguistic formulation which, in this case, is not relevant];

The final algebraic formulation proposed by group 1 is: $x^2+y^2+2xy-1=0$, which in natural language is interpreted as follows: «A circumference» [the interpretation error is evident, but we decide to pass over]; we carry out the transformation which leads to: $x+y=\frac{1}{x+y}$ that after a few attempts is interpreted as «A sum that has the same value of its reciprocal»;

question: «But $x+y=\frac{1}{x+y}$ is it or not the "circumference" we started with?»;

student A: «Absolutely no, a circumference must have x^2+y^2»;

student B: «If we simplify, yes».

One can ask whether or not it is the transformation that gives a *sense*: from the episode it seems that if one would perform the inverse passages, then one would return to a "circumference". But it could also instead be that the meanings are attributed to the specific representations, without links between them, as if the transformation that makes sense for the teacher it does not make sense for the person who performs it.

Example 2

The text written in natural language requires the algebraic writing of the sum of three consecutive natural numbers and the proposal of group II is: $(n-1)+n+(n+1)$ [obviously the doubt remains in the case of $n=0$, but we decide to pass over]; we carry out the transformation that leads to the following writing: $3n$ that is interpreted as: «The triple of a natural number»;

question: «But $3n$ can be thought as the sum of three consecutive natural numbers?»;

student C: «No, *like this* no, *like this* it is the sum of three equal numbers, that is n».

Example 3

We consider the sum of the first 100 natural positive numbers: 1+2+…+99+100; we perform Gauss classical transformation; 101×50; this representation is recognized as the solution of the problem but not as the representation of the starting object; the presence of the multiplication sign compels all the students to look for a sense in mathematical objects in which the "multiplication" term (or similar terms) appears;

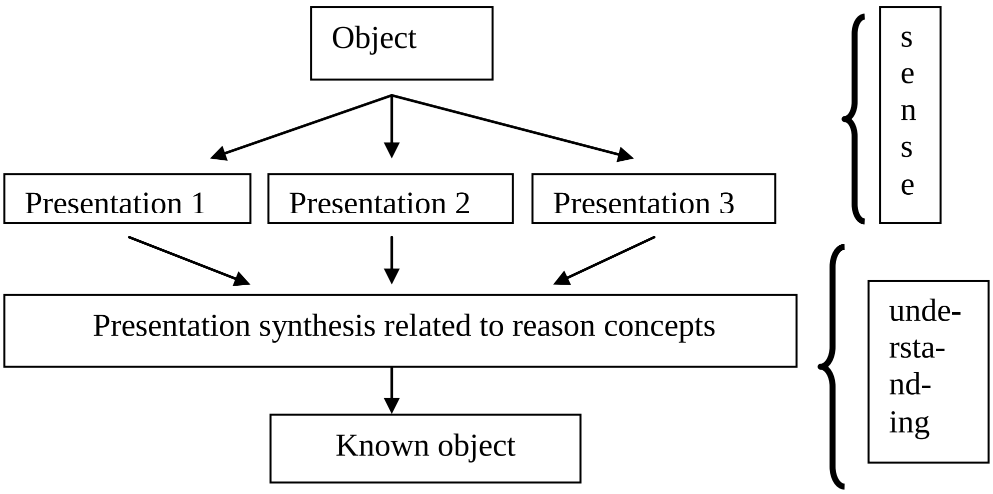
question: «But 101×50 is it or not the sum of the first 100 positive natural numbers?»;

student D: «That one, is not a sum, that is a multiplication; it corresponds to the sum, but it is not the sum».

In these episodes we witness a constant change of meaning during the transformations: each new representation has a specific meaning of its own not referable to the one of the starting representations, even if the passage from the first to the second ones has been performed in an evident and shared manner.

The causes of the changes of meaning

What are the causes of the changes of meaning, what origin do they have?
We can start from this diagram that we appreciate a lot because of its attempt to put in the right place the ideas of *sense* and *understanding* (Radford, 2004a).



The process of meanings endowment moves at the same time within various semiotic systems, simultaneously activated; we are not dealing with a pure classical dichotomy: treatment/conversion leaves the meaning prisoner of the internal semiotic structure, but with something much more complex. Ideally, from a structural point of view, the meaning should come from within the semiotic system we are immersed in. Therefore, in *Example 2,* the pure passage from $(n\text{-}1)+n+(n+1)$ to $3n$ should enter the category: treatment semiotic transformation. But what happens in the classroom practice, and not only with novices in algebra, is different. There is a whole path to cover, starting from single specific meanings culturally endowed to the signs of the algebraic language ($3n$ is the triple of something; 101×50 is a product, not a sum). Thus, there are sources of meanings relative to the algebraic language that anchor to meanings culturally constructed, previously in time; such meanings often have to do with the arithmetic language. From an, so to speak, "external" point of view, we can trace back to seeing the different algebraic writings as equally significant since they are obtainable through semiotic treatment, but from inside this picture is almost impossible, bound as it is to the culture constructed by the individual in time. In other words, we can say that students (not only novices) turn out bridled to sources of meaning that cannot be simply governed by the syntax of the algebraic language. Each passage gives rise to forms or symbols to which a specific meaning is recognised because of the cultural processes THROUGH which it has been introduced.

In Luis Radford's semiotic anthropological approach (ASA) mathematical knowledge is seen as the product of a reflexive cognitive mediated praxis. «Knowledge as cognitive praxis (*praxis cogitans*) underlines the fact that what we know and the way we come to know it are underpinned by ontological positions and by cultural processes of meaning production that give form to a certain way of rationality within which certain types of questions and problems are posed. The *reflexive* nature of knowledge must be understood in Ilyenkov's sense, that is, as a distinctive component that makes cognition an intellectual reflexion of the external world in accordance with the forms of individuals' activity (Ilyenkov, 1977, page 252). The mediated nature of knowledge refers to the role played by tools and signs as means of knowledge objectification and as instruments that allow to bring to a conclusion the cognitive *praxis*» (Radford, 2004b, page 17).

On the other hand, «the object of knowledge is not filtered only by our senses, as it appears in Kant, but overall by the cultural modes of signification (...). (...) the object of knowledge is filtered by the technology of the semiotic activity. (...) knowledge is culturally mediated» (Radford, 2004b, page 20). «(…) These terms are the semiotic means of objectification. Thanks to these means, the general object that always remains directly inaccessible starts to take form: it starts to become an "object of consciousness" for the pupils. Although general, these objects however remain *contextual*» (Radford, 2004b, page 23).

The approach to the object and its appropriation on the part of the individual who learns, are the result of personal intentions with which individuals express themselves through experiences that see the objects used in suitable contexts: «Intentions occur in contextual experiences that Husserl called *noesis*. The conceptual content of such experiences he termed *noema*. Thus, noema corresponds to the way objects are grasped and become known by the individuals while noesis relates to the modes of cultural categorical experiences accounting for the way objects become attended and disclosed (Husserl, 1931)» (Radford, 2002, page. 82).

In the cases we presented above, and in mathematics in general, it is clear that the objects are attended from the first moment in their formal expression, in our case in the algebraic language; the individual learns to formally handle these signs, but what happens to the initial mathematical object? What happens to the initial meanings? We suppose that these meanings are tightly bound to the arithmetic experience of the pupil and overall to the way in which such an experience becomes objective through its objective transposition into ordinary language. Deep understanding of algebraic or, in general, formal manipulation, holds a prominent position.

Through an interesting comparison, Radford expresses himself on this point as follows: «While Russell (1976, page 218) considered the formal manipulations of signs as empty descriptions of reality, Husserl stressed the fact that such a manipulation of signs requires a shift of intention, a noematic change: the focus becomes the signs themselves, but not as signs *per se*. And he insisted that the abstract manipulation of signs is supported by new meanings arising from rules resembling the rules of a game (Husserl

1961, page 79), which led him talk about signs having a *game signification* (...)» (Radford, 2002, page 88).

After having shown the broad and complex significance of the phenomenon, we must refer to other disciplines in order to understand better and better the issue of the different meanings of algebraic expressions, that is, in order to give a significant contribution to this aspect of mathematics education.

Analysis of the phenomenon thanks to theories "external" of mathematics education

We believe that some theories "external" of mathematics education can have, and in fact they already have, a strong influence on the analyses of various phenomena, like the ones described here, therefore giving a contribution to changing the theoretical frame of our discipline in its future research developments.

Philosophy. In section 2, we have seen how philosophy (Husserl's phenomenology) can have remarkable contribution and we will not repeat ourselves.

Learning is taking consciousness of a general object in accordance with the modes of rationality of the culture one belongs to.

More importantly we must face here the issue of the philosophical dilemma on concept and object, and even more the problem of the need of a previous choice between realist and pragmatist positions (D'Amore, Fandiño Pinilla, 2001; D'Amore, 2003; D'Amore, 2007).

In **realist theories** the meaning is a «conventional relationship between signs and ideal or concrete entities that exist independently of linguistic signs; they therefore suppose a conceptual realism» (Godino and Batanero, 1994). As Kutschera (1979) already claimed: «According to this conception the meaning of a linguistic expression does not depend on its use in concrete situations, but it happens that the use holds on meaning, since a clear distinction between pragmatics and semantics is possible».

In the realist semantic that it derives, we attribute to linguistic expressions purely semantic functions; the meaning of a proper name (as: 'Bertrand Russell') is the object that such proper name indicates (in such a case: Bertrand Russell); the individual statements (as: 'A is a river') express facts that describe reality (in such a case; A is the name of a river); the binary predicates (as: 'A reads B') designate attributes, those indicated by the phrase that expresses them (in this case: person A reads thing B). Therefore every linguistic expression is an attribute of certain entities: the nominal relationship that derives is the only semantic function of expressions.

We recognise here the bases of Frege's, Carnap's and Wittgenstein's (*Tractatus)* positions.
A consequence of this position is the acknowledgement of a "scientific" observation (at the same time therefore, empiric and subjective or intersubjective) as it could be, at a first level, a statement and predicate logic.

From the point of view we are mostly interested in, if we apply to Mathematics the ontological assumption of realist semantics, we necessarily draw a platonic picture of mathematical objects: notions, structures, etc. have a real existence that does not depend on human being, as they belong to an ideal domain; "to know" from a mathematical point of view means "to discover" in such domain entities and relationships between them. It is also obvious that such picture implies an absolutism of mathematical knowledge, since it is thought as a system of external certain truths that cannot be modified by human experience because they precede or, at least, are extraneous and independent from it.

Akin positions, although with different nuances, were sustained by Frege, Russell, Cantor, Bernays, Goedel,…; they also encountered violent criticisms [Wittgensteins' *Conventionalism* and Lakatos' *quasi-empirism* : see Ernest (1991) and Speranza (1997)].

In **pragmatic theories** linguistic expressions have different meanings according to the context in which they are used and therefore any scientific observation is impossible, since the only possible analysis is a "personal" and subjective one, anyway circumstantial and not generalizable. We cannot but analyse the different "uses": the set of "uses" in fact determines the meaning of objects.

We recognize here Wittgenstein's positions of the *Philosophical Investigations,* when he admits that the significance of a word depends on its function in a "linguistic game", since in such game it has a way of 'use' and a concrete purpose for which it has been precisely used: therefore the word does not have a meaning *per se*, but nevertheless, it can be meaningful.

Mathematical objects are therefore symbols of cultural units that emerge from a system of uses that characterise human pragmatics (or at least of individuals' homogeneous groups) and that continuously modify in time, also according to needs. In fact, mathematical objects and the meaning of such objects depend on the problems that we face in Mathematics and on their solution processes.

	"REALIST" THEORIES	"PRAGMATIC" THEORIES
meaning	conventional relationship between signs and concrete or ideal entities independent of linguistic signs	depends on the context and use
semantics Vs pragmatics	clear distinction	no distinction or faded distinction
objectivity or intersubjectivity	complete	missing or questionable
semantics	linguistic expressions have purely semantic functions	linguistic expressions and words have "personal" meanings, are meaningful in suitable contexts, but they don't have absolute meanings *per se*
analysis	possible and licit: logic for example	only a "personal" or subjective analysis is possible, not generalizable, not absolute
consequent epistemological picture	platonic conception of mathematical objects	problematic conception of mathematical objects
to know	to discover	to use in suitable contexts.
knowledge	is an absolute	is relative to circumstance and specific use
examples	Wittgenstein in *Tractatus*, Frege, Carnap [Russell, Cantor, Bernays, Gödel]	Wittgenstein in *Philosophical Invesigations* [Lakatos]

It is obvious and it would be easy to prove with philosophical examples, that the two fields are not fully complementary and clearly separated even if, for reasons of clarity, we preferred giving this "strong" impression.

With regard to the philosophical bases of mathematics education, we have decided to stay in the pragmatic domain that seems much closer to the reality of the empiric process of Mathematics teaching/learning. It seems that each specification that appears in the right column, cell by cell, is part of the same process and of its explicitation. It seems that focusing didactic activity (and therefore research) on learning and consequently on epistemology of the domain that has the student as a protagonist, we are obliged to interpret each step of knowledge construction as responding to the *language game*, therefore admitting that the semantics blur the use pragmatics.

Sociology. In D'Amore (2005) and D'Amore and Godino (2007), we show how the results of the analyses relative to the behaviours of individuals engaged in an activity of conceptual learning of mathematical objects, their transformations of the descriptions of such objects from ordinary language to formal language, the manipulations of such formalizations can be framed within a sociological interpretation key: the learning environment is framed within a sociological interpretation key and the individuals' behaviours are interpreted through the notion of "practice" and its "meta-practice" evolution. Essentially the individuals shift from a shared practice, recognized as characteristic of the social group they belong to, to a meta-practice that modifies such characteristic; the interpretative behaviour therefore ceases to be global and social and becomes local and personal; the notions that come into play in such interpretations are specific of the circumstance and not of the situation in its entirety.

We pass over this point, referring back to the quoted texts.

Anthropology. In D'Amore and Godino (2006, 2007) we go into strongly anthropological details in order to explain the nature of the choices of the individual who learns mathematics. In such articles we highlight how «Having obliged the researcher to point all his attention to the activities of human beings who have to do with mathematics (not only solving problems, but also communicating mathematics) is one of the merits of the anthropological point of view, inspiring other points of view, amongst which the one that today we call "anthropological" in the proper sense: the ATD, anthropological theory of didactics (of mathematics) (Chevallard, 1999; page 221). Why this adjective "anthropological"? It is not an exclusiveness of the approach created by Chevallard in 80s, as he himself declares (Chevallard, 1999), but an "effect of the language" (page 222); it distinguishes the theory, identifies it, but it is not peculiar to such theory in a univocal way» (D'Amore and Godino, 2006, page 15). The ATD is almost exclusively centred on the institutional dimension of mathematical knowledge, as a development of the research program started with fundamental didactics. The crucial point is that «ATD places the *mathematical* activity, and therefore *the study* in mathematics activity, *in the set of human activities and of social institutions*» (Chevallard, 1999).
This kind of analyses, although subjected to criticisms in D'Amore and Godino (2006, 2007), has opened the way to the use of anthropology as a critical instrument, as a new theoretical frame at research into mathematics education, in accordance with what has been already highlighted in the above quoted articles. It is the human being, strong of the acquired culture, strong of the specific expressive, communicative luggage, who handles formal writings and gives them a meaning that it cannot be anything else but coherent with his social history; every meaning of each formal expression is the result of an anthropological comparison between a lived history and a here-and- now that must be coherent with that history.

We pass over this point, referring back to the quoted texts.

Psychology. In D'Amore and Godino (2006) we show how the shift from the anthropological picture to the onto-semiotic one is made necessary (amongst other things) by the need of not trivializing the presence of psychology in the study of learning and, in general, classroom situations. In D'Amore (1999) we show, for example, how ideas on representation drawn from psychology, regarding the explanation of the passage from image (weak) to model (stable) of concepts (Paivio, 1971; Kosslyn, 1980; Johnson-Laird, 1983; Vecchio, 1992), can be placed as a unitary basis of the explanation of several didactic phenomena, as intuitive models, the shift from internal to external models, the figural concepts, up to misconceptions, studied mainly in the 80s. Also the ideas of frame and script (Bateson, 1972; Schank and Abelson, 1977) have been used for the same purpose.

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